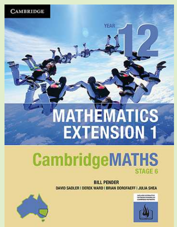
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1
YEAR 12 COURSE

Topic summary and exercises:

With references to



X1 X2 **Differential Equations**

Name: ....

Initial version by H. Lam, April 2020. Last updated November 21, 2021.
Various corrections by students and members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

- (!) Beware! Heed warning.
- (R) Revision content.
- (X1) Mathematics Extension 1 content.
- (X2) Mathematics Extension 2 content.
- (L) Literacy: note new word/phrase.
- (E) Extension content: unlikely to be in the syllabus and therefore not examinable.

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME12-4 uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution

ME12-6 chooses and uses appropriate technology to solve problems in a range of contexts

Syllabus subtopics

ME-C2 Further Calculus Skills

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Ordinary Differential Equations

Learning Goal(s)

Knowledge	Skills	Understanding
How to recognise	Recognise	ODEs and non unique solutions to ODEs

By the end of this section am I able to:

29.1 Recognise that an equation involving a derivative is called a differential equation

29.2 Recognise that solutions to differential equations are functions and that these solutions may not be unique

1.1 Definitions and Rationale

Definition 1

An *ordinary differential equation* (ODE) is an equation that contains terms of $y = f(x)$ and ......derivatives...... of $f(x)$.

Abbreviated to ...'DE'... in high school textbooks.

Important note

Why 'ordinary'? Later in STEM courses at university, *partial differential equations* (PDEs) will be studied. These involve *partial* derivatives, e.g.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

(The PDE above is *Laplace's Equation*, arises in heat and diffusion)

1.1.1 Order

Definition 2

The **order** of an ODE is the order of the ......highest...... ......derivative......

Laws/Results

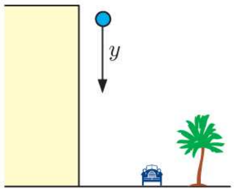
The number of ......arbitrary...... ......constants...... arising from an ODE is the same as the ODE's order.

1.1.2 Some simple applications of ODEs

(Haese, Haese, & Humphries, 2017, Section 8C, p.237)

- Derivatives in this topic are written as y' or $\frac{dy}{dx}$, instead of $f'(x)$ or $\dot{x}$.

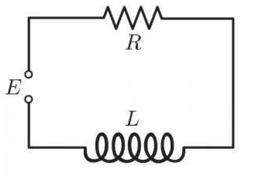
A falling object X1



$$\frac{d^2y}{dt^2} = 9.8$$

- Order: 2

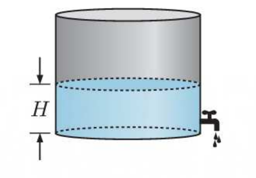
Current in an RL circuit X1



$$L\frac{dI}{dt} + RI = E$$

- Order: 1

Water leaking from a tank X1



$$\frac{dH}{dt} = -a\sqrt{H}$$

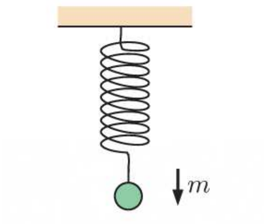
- Order: 1

A parachutist X2

$$m\frac{dv}{dt} = mg - kv^2$$

- Order: 1

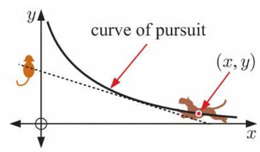
Object on a spring X2



$$m\frac{d^2y}{dt^2} = -ky$$

- Order: 2

A dog pursuing a cat



$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

- Order: 2

Important note

One notable example not shown above for (x1) : the *logistic curve*. See Section 2.3 on page 24.

1.2 Solutions to ODEs

Definition 3

A **solution** to an ODE is an equation of a ......relation...... or ......function......

Important note

(!) Equations dealt with during Years 7-12, had $x = \{ \text{ value } \}$.

- Definitions 4 and 5 on the next page provide further insight.

1.2.1 Verifying solutions

- (R) See also **Topic 10 - Rates of change**, Section 2.

Steps

1. ......Differentiate...... the solution provided as many times as the ......order...... of the ODE.

2. ......Substitute...... the required derivatives into the ODE.

- ......Mask...... any functions of x with y', y'' etc.

Example 1

[2015 VCE Specialist Mathematics Paper 2 Q14] A differential equation that has $y = x \sin x$ as a solution is

(A) $\dfrac{d^2y}{dx^2} + y = 0$ (C) $\dfrac{d^2y}{dx^2} + y = -\sin x$ (E) $\dfrac{d^2y}{dx^2} + y = 2\cos x$

(B) $x\dfrac{d^2y}{dx^2} + y = 0$ (D) $\dfrac{d^2y}{dx^2} + y = -2\cos x$

1.2.2 Types of solutions

Definition 4

Ⓛ The **general solution** to a differential equation involves ....arbitrary........ constants.

- The general solution is a concise way to represent ....infinitely........ many solutions.
- The ....indefinite........ ....integral........ is one type of general solution.

Definition 5

Ⓛ The **particular solution** to a differential equation involves substituting ....initial.... ....values.... into the general solution.

Often abbreviated to y_P.

A question that involves finding a particular solution, is known as an *initial value problem* (IVP).

Example 2

[Section 8C] (Haese et al., 2017) Consider the differential equation

$$\frac{dy}{dx} - 3y = 3$$

(a) Show that $y = ce^{3x} - 1$ is a solution to the differential equation for any constant c.

(b) Sketch the solution curves for $c = 0, \pm 1, \pm 2, \pm 3$.

(c) Find the particular solution which passes through $(0, 2)$

(d) Find the equation of the tangent to the particular solution at $(0, 2)$.

Answer: $y_P = 3e^{3x} - 1$, tangent: $y = 9x + 2$

Example 3

[2011 Ext 2 HSC Q4] A mass is attached to a spring and moves in a resistive medium. The motion of the mass satisfies the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

where y is the displacement of the mass at time t.

i. Show that if $y = f(t)$ and $y = g(t)$ are both solutions to the differential equation and A and B are constants, then **2**

$$y = Af(t) + Bg(t)$$

is also a solution.

ii. A solution of the differential equation is given by $y = e^{kt}$ for some values of k, where k is a constant. **2**

Show that the only possible values of k are $k = -1$ and $k = -2$.

iii. A solution of the differential equation is **3**

$$y = Ae^{-2t} + Be^{-t}$$

When $t = 0$, it is given that $y = 0$ and $\dfrac{dy}{dt} = 1$.

Find the values of A and B.

Answer: $A = -1$, $B = 1$

1.2.3 Additional exercises

Source Haese et al. (2017, Ex 8C)

1. Verify that:

(a) $y = x^4$ is a solution to $\dfrac{dy}{dx} = 4x^3$

(b) $y = 5e^{2x}$ is a solution to $\dfrac{dy}{dx} = 2y$

(c) $y = \sqrt{x^2+1}$ is a solution to $\dfrac{dy}{dx} = \dfrac{x}{y}$

(d) $y = -\dfrac{1}{x}$ is a solution to $\dfrac{dy}{dx} = y^2$

(e) $y = 3e^{\frac{x^2}{2}+x}$ is a solution to $\dfrac{dy}{dx} - y = xy$

(f) $y = x^3 + C$ is the general solution to $\dfrac{dy}{dx} = 3x^2$

(g) $y = Ce^{-x}$ is the general solution to $\dfrac{dy}{dx} = -y$

(h) $y = -\dfrac{2}{x^2+C}$ is the general solution to $\dfrac{dy}{dx} = xy^2$

2. Consider the differential equation $\dfrac{dy}{dx} = 4x$.

(a) Show that $y = 2x^2 + C$ is a solution to the differential equation for any constant C.

(b) Sketch the solution curves for $C = 0, \pm 1, \pm 2$.

(c) Find the particular solution which passes through $\left(1, \frac{1}{2}\right)$.

(d) Find the equation of the tangent to the particular solution at $\left(1, \frac{1}{2}\right)$.

3. Consider the differential equation $\dfrac{dy}{dx} = 2x - y$.

(a) Show that $y = 2x - 2 + Ce^{-x}$ is a solution to the differential equation for any constant C.

(b) Sketch the solution curves for $C = 0, \pm 1, \pm 2$.

(c) Find the particular solution which passes through $(0, 1)$.

(d) Find the equation of the tangent to the particular solution at $(0, 1)$.

Answers

2. (c) $y = 2x^2 - \frac{3}{2}$ (d) $y = 4x - \frac{7}{2}$ **3.** (c) $y = 2x - 2 + 3e^{-x}$ (d) $y = -x + 1$

Section 2

First order ODEs

Learning Goal(s)

Knowledge	Skills	Understanding
First order ODEs	Solve	Features of first order ODEs and exponentials

By the end of this section am I able to:

29.4 Solve simple first-order differential equations

29.5 Recognise the features of a first-order linear differential equation and that exponential growth and decay models are first-order linear differential equations, with known solutions

2.1 Linear

Definition 6

A *first order linear* ODE take the form

$$y' + f(x)y = g(x)$$

Special cases of the first order linear ODE

- Simple integration: $f(x) = 0$. See Section 2.1.1 on the following page
- Change of subject: $f(x) = k$ (non zero constant) and $g(x) = 0$. See Section 2.1.2 on page 15.
- (E) Integrating factor: first year university. Multiply throughout by an integrating factor

$$I = e^{\int f(x)\,dx}$$

and use the product/chain rules:

$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + f(x)Iy$$

2.1.1 Simple integration

Steps

For equations of the form $y' = f(x)$

1. Evaluate the ......indefinite...... ......integral......

$$y = \int f(x)\,dx$$

2. Substitute any ......initial...... ......values...... where appropriate.

- No further examples are provided here.
- Most of these are reviewing integration techniques from **Topic 27 - Further Integration** and other calculus based topics prior to this.

Further exercises

Ex 13A

- Q1-16

2.1.2 Change of subject

Steps

For equations of the form $y' = g(y)$

1. Rewrite in differential form: $\frac{dy}{dx} = g(y)$
2. Gather y with dy, and change the subject to dx
3. Evaluate the indefinite integral on both sides.

$$\int \frac{dy}{g(y)} = \int dx$$

4. Substitute any initial values where appropriate.

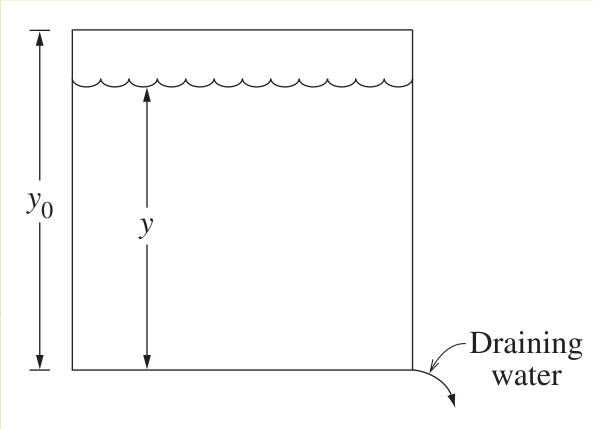
Example 4

Solve: $\frac{dy}{dx} = \frac{1}{y^2}$.

Answer: $y = \sqrt[3]{3x + C},\ C \in \mathbb{R}$

Example 5

[2002 Ext 2 HSC Q7] The diagram represents a vertical cylindrical water cooler of constant cross-sectional area A. Water drains through a hole at the bottom of the cooler.



From physical principles, it is known that the volume V of water decreases at a rate given by

$$\frac{dV}{dt} = -k\sqrt{y}$$

where k is a positive constant and y is the depth of water.

Initially the cooler is full and it takes T seconds to drain. Thus $y = y_0$ when $t = 0$, and $y = 0$ when $t = T$.

i. Show that $\dfrac{dy}{dt} = -\dfrac{k}{A}\sqrt{y}$. **1**

ii. By considering the equation for $\dfrac{dt}{dy}$, or otherwise, show that **4**

$$y = y_0\left(1 - \frac{t}{T}\right)^2 \quad \text{for } 0 \le t \le T$$

iii. Suppose it takes 10 seconds for half the water to drain out. How long does it take to empty the full cooler? **2**

Answer: $T = 10\left(2+\sqrt{2}\right)$ seconds

2.2 Separable

Also known as *separation of variables*.

Steps

For equations of the form $y' = f(x)g(y)$.

1. Rewrite in differential form: $\frac{dy}{dx}$ $= f(x)g(y)$

2. Gather $g(y)$ with dy, and $f(x)$ with dx.

3. Evaluate the indefinite integral on both sides.

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

4. Substitute any initial values where appropriate.

Important note

Ⓡ The following ODEs from **Topic 10 - Rates of change**, Section 2 have exponential solutions:

- $\frac{dy}{dx} = ky$
- $\frac{dy}{dx} = k(y - a)$

Example 6

(a) Solve $y' = -2xe^y$

(b) Find the solution curve through $(0, 0)$.

Answer: (a) $y = -\ln\left(x^2 + C\right)$ (b) $y = -\ln\left(x^2 + 1\right)$

Example 7

(a) Solve $y' = -xy^2$

(b) Find the particular solution given that:

i. $y(1) = \frac{1}{2}$

ii. $y(2) = 0$

Answer: i. $y = \frac{2}{x^2+3}$ or $y = 0$ ii. $y = 0$ is also possible.

Important note

Always check whether any constant functions $y = k$ are solutions of the ODE.

Example 8

[2019 VCE Specialist Mathematics Paper 1 Q1] (4 marks) Solve the differential equation

$$\frac{dy}{dx} = \frac{2ye^{2x}}{1+e^{2x}}$$

given that $y(0) = \pi$.

Answer: $\frac{\pi}{2}\left(1+e^{2x}\right)$

Example 9

[2016 VCE Specialist Mathematics Paper 1 Q9] (5 marks) Solve the differential equation

$$\left(\sqrt{2-x^2}\right)\frac{dy}{dx} = \frac{1}{2-y}$$

given that $y(1) = 0$. Express y as a function of x. **Answer:** $y = 2 - \sqrt{4 + \frac{\pi}{2} - 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)}$

Further exercises

Ex 13C

- Q1-15

2.2.1 Additional exercises

Source Haese et al. (2017, Ex 8E).

1. Solve the following separable differential equations.

(a) $\dfrac{dy}{dx} = \dfrac{x}{y^2}$

(b) $\dfrac{dy}{dx} = \dfrac{2x}{e^y}$

(c) $\dfrac{dy}{dx} = 3xy$

(d) $\dfrac{dy}{dx} = 2x\sqrt{y}$

(e) $\dfrac{dy}{dx} = y \sin x$

(f) $\dfrac{dy}{dx} = -x\sqrt{y+1}$

(g) $\dfrac{dy}{dx} = \dfrac{y}{x}$

(h) $\dfrac{dy}{dx} = 3x^2 e^y$

(i) $\dfrac{dy}{dx} = \dfrac{y+2}{x-1}$

2. Solve:

(a) $\dfrac{dy}{dx} = y$

(b) $\dfrac{dy}{dx} = \dfrac{1}{y}$

(c) $\dfrac{dy}{dx} = y - 4$

(d) $\dfrac{dP}{dt} = 3\sqrt{P}$

(e) $\dfrac{dQ}{dt} = 2Q + 3$

(f) $\dfrac{dQ}{dt} = \dfrac{1}{2Q+3}$

3. Solve:

(a) $\dfrac{dy}{dx} = \dfrac{y}{x^2+1}$

(b) $4 + \dfrac{dy}{dx} = 2y$

(c) $\left(x^2+5\right)\dfrac{dy}{dx} = \dfrac{2x}{y^2}$

(d) $\left(\sqrt{4-x^2}\right)\dfrac{dy}{dx} = 1 - y$

(e) $\dfrac{dy}{dx} = xy^2 - 2y^2$

(f) $y\dfrac{dy}{dx} = \dfrac{6x\sqrt{y}}{x^2+5}$

4. Find the particular solution to:

(a) $\dfrac{dy}{dx} = \dfrac{3x}{y^2}$ given that $y(0) = 1$

(b) $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{3}$ given that $y(44) = 9$

(c) $\dfrac{dy}{dx} = y + yx^2$ given that $y(0) = 1$

(d) $\dfrac{dy}{dx} = \dfrac{3x}{\cos y}$ given that $y(1) = 0$

(e) $e^y\left(2x^2+4x+1\right)\dfrac{dy}{dx} = (x+1)\left(e^y+3\right)$ given that $y(0) = 2$

(f) $x\dfrac{dy}{dx} = \cos^2 y$ given that $y(e) = \frac{\pi}{4}$

5. (a) Show that $\dfrac{3-x}{x^2-1} = \dfrac{1}{x-1} - \dfrac{2}{x+1}$.

(b) Find the particular solution to $\dfrac{dy}{dx} = \dfrac{3y - xy}{x^2-1}$ given that $y(0) = 1$.

6. (a) Show that $\dfrac{5x+4}{x^2+x-2} = \dfrac{2}{x+2} + \dfrac{3}{x-1}$.

(b) Find the particular solution to $\dfrac{dy}{dx} = \dfrac{5xy^2 + 4y^2}{x^2+x-2}$ given that $y(0) = -\frac{1}{2}$.

7. (a) Show that $\dfrac{2}{x^2-1} = \dfrac{1}{x-1} - \dfrac{1}{x+1}$.

(b) Find the general solution to $\dfrac{dy}{dx} = \dfrac{x^2y + y}{x^2-1}$.

Answers

1. (a) $y = \sqrt[3]{\frac{3}{2}x^2 + C}$ (b) $y = \ln\left(x^2 + C\right)$ (c) $y = Ae^{\frac{3}{2}x^2}$ (d) $y = \left(\frac{x^2}{2} + C\right)$ (e) $y = Ae^{-\cos x}$ (f) $y = \left(-\frac{1}{4}x^2 + C\right)^2$ (g) $y = Ax$ (h) $y = -\ln\left(C - x^3\right)$ (i) $y = A(x-1) - 2$ **2.** (a) $y = Ae^x$ (b) $y = \pm\sqrt{2x + C}$ (c) $y = Ae^t + 4$ (d) $P = \left(\frac{3}{2}t + C\right)^2$ (e) $Q = Ae^t - \frac{3}{2}$ (f) $t = Q^2 + 3Q + C$ **3.** (a) $y = Ae^{\tan^{-1} x}$ (b) $y = Ae^{2x} + 2$ (c) $y = \sqrt[3]{3\ln\left(x^2+5\right) + C}$ (d) $y = 1 + Ae^{-\sin^{-1}\left(\frac{x}{2}\right)}$ (e) $y = \frac{1}{-\frac{1}{2}x^2 + 2x + C}$ (f) $y = \left(\frac{9}{2}\ln\left(x^2+5\right) + C\right)^{\frac{2}{3}}$ **4.** (a) $y = \sqrt[3]{\frac{9}{2}x^2 + 1}$ (b) $y = \frac{1}{36}(x-26)^2$ (c) $y = e^{x + \frac{1}{3}x^3}$ (d) $y = \sin^{-1}\left(\frac{3}{2}x^2 - \frac{3}{2}\right)$ (e) $y = \ln\left[\sqrt[4]{|2x^2 + 4x + 1|}\left(e^2 + 3\right) - 3\right]$ (f) $y = \tan^{-1}\left(\ln|x|\right)$ **5.** $y = \frac{1-x}{(x+1)^2}$ **6.** $y = -\frac{1}{\ln\left|\frac{(x+2)^2(x-1)^3}{4}\right| + 2}$ **7.** $y = Ae^x\left(\frac{x-1}{x+1}\right)$

2.3 The logistic curve

Learning Goal(s)

Knowledge
Logistic curve

Skills
Solve

Understanding
Chemistry/Biology/Economics phenomena modelled by the logistic curve

By the end of this section am I able to:

29.6 Model and solve differential equations including to the logistic equation that will arise in situations where rates are involved, for example in chemistry, biology and economics

2.3.1 History and background

Ⓔ Other information: `https://en.wikipedia.org/wiki/Logistic_function`

History

Pierre-François Verhulst was born in 1804 in Brussels. He obtained a PhD in mathematics from the University of Ghent in 1825. He was also interested in politics.

While in Italy to contain his tuberculosis, he pleaded without success in favour of a constitution for the Papal States. After the revolution of 1830 and the independence of Belgium, he published a historical essay on an eighteenth century patriot. In 1835 he became professor of mathematics at the newly created Free University in Brussels.

In 1838, Verhulst published a *Note on the law of population growth*:

> We know that the famous Malthus showed the principle that the human population tends to grow in a geometric progression so as to double after a certain period of time, for example every twenty five years. This proposition is beyond dispute if abstraction is made of the increasing difficulty to find food [...]
>
> The virtual increase of the population is therefore limited by the size and the fertility of the country. As a result the population gets closer and closer to a steady state.

Photo and text: Bacaër (2011, p. 35-36)

Watch: `https://www.youtube.com/watch?v=C_3VV01wzpk`

(E) Other information: `https://en.wikipedia.org/wiki/Logistic_function`

Fill in the spaces

- The *logistic curve* is due to ......Verhulst...... (1838)

$$P(t) = \frac{1}{1 + Ae^{-rt}}$$

- Models population growth where population ......regulate...... themselves:
  - Initially, population *increases* rapidly
  - Competition for ...food... / ...space... / ...resources... pushes the population to a natural ...limit... .
- Many applications:
  - Growth of ...tumour...
  - ...Economics...
  - ...Social... ...media...
- Find the derivative w.r.t. t, then rewrite in terms of P:

$$\frac{dP}{dt} = rP(1 - P)$$

  - Derivative ...proportional... to P *and* $(1 - P)$
  - For a small population, $\frac{dP}{dt} \approx rP$
  - As time increases, $\frac{dP}{dt} \approx 0$
- Graph of $\frac{dP}{dt}$ against P:

2.3.2 Definition

Definition 7

The logistic curve takes on the form

$$\frac{dN}{dt} = kN\left(P - N\right)$$

where P and k are constants.

Example 10

[1992 3U HSC Q5] In a flock of 1 000 chickens, the number P infected with a disease at time t **years** is given by

$$P = \frac{1\,000}{1 + ce^{-1\,000t}} \qquad \text{where } c \text{ is a constant}$$

(i) Show that, eventually, all the chickens will be infected. **1**

(ii) Suppose that when time $t = 0$, exactly one chicken was infected. After how many **days** will 500 chickens be infected? **2**

(iii) Show that $\dfrac{dP}{dt} = P(1\,000 - P)$. **2**

Example 11

[2008 Ext 2 HSC Q5] A model for the population, P, of elephants in Serengeti National Park is

$$P = \frac{21\,000}{7 + 3e^{-\frac{t}{3}}}$$

where t is the time in years from today.

i. Show that P satisfies the differential equation **2**

$$\frac{dP}{dt} = \frac{1}{3}\left(1 - \frac{P}{3\,000}\right)P$$

ii. What is the population today? **1**

iii. What does the model predict that the eventual population will be? **1**

iv. What is the annual percentage rate of growth today? **1**

Example 12

[**2016 2U HSC Q16**] Some yabbies are introduced into a small dam. The size of the population, y, of yabbies can be modelled by the function

$$y = \frac{200}{1 + 19e^{-0.5t}}$$

where t is the time in months after the yabbies are introduced into the dam.

i. Show that the rate of growth of the size of the population is **2**

$$\frac{1\,900e^{-0.5t}}{\left(1 + 19e^{-0.5t}\right)^2}$$

ii. Find the range of the function y, justifying your answer. **2**

iii. Show that the rate of growth of the size of the population can be rewritten as **1**

$$\frac{y}{400}(200 - y)$$

iv. Hence, find the size of the population when it is growing at its fastest rate. **2**

Answer: (i) Show (ii) $R = \{y : 10 \leq y < 200\}$ (iii) Show (iv) $y = 100$

Example 13

[2010 Ext 2 HSC Q5]

(b) (X2) Show that **2**

$$\int \frac{dy}{y(1-y)} = \ln\left(\frac{y}{1-y}\right) + c$$

for some constant c, where $0 < y < 1$.

(c) A TV channel has estimated that if it spends $\$x$ on advertising a particular program it will attract a proportion $y(x)$ of the potential audience for the program, where

$$\frac{dy}{dx} = ay(1-y)$$

and $a > 0$ is a given constant.

i. Explain why $\dfrac{dy}{dx}$ has its maximum value when $y = \dfrac{1}{2}$. **1**

ii. Using part (b), or otherwise, deduce that **3**

$$y(x) = \frac{1}{ke^{-ax}+1}$$

for some constant $k > 0$

iii. The TV channel knows that if it spends no money on advertising the program then the audience will be one-tenth of the potential audience. **1**

Find the value of the constant k referred to in part (c)ii.

iv. What feature of the graph $y = \dfrac{1}{ke^{-ax}+1}$ is determined by the result in part (c)i? **1**

v. Sketch the graph $y = \dfrac{1}{ke^{-ax}+1}$. **1**

Example 14

[2020 Ext 1 HSC Sample Q14] The population of foxes on an island is modelled by the logistic equation

$$\frac{dy}{dt} = y(1-y)$$

where y is the fraction of the island's carrying capacity reached after t years.

At time $t = 0$, the population of foxes is estimated to be one-quarter of the island's carrying capacity.

i. Use the substitution $y = \frac{1}{1-w}$ to transform the logistic equation to $\frac{dw}{dt} = -w$. **2**

ii. Using the solution of $\frac{dw}{dt} = -w$, find the solution of the logistic equation for y satisfying the initial conditions. **2**

iii. How long will it take for the fox population to reach three-quarters of the island's carrying capacity? **2**

Answer: $t = \ln 9$ years

Example 15

[2021 Ext 1 HSC Q14] (4 marks) In a certain country, the population of deer was estimated in 1980 to be 150 000. The population growth is given by the logistic equation $\dfrac{dP}{dt} = 0.1P\left(\dfrac{C-P}{C}\right)$ where t is the number of years after 1980 and C is the carrying capacity.

In the year 2000, the population of deer was estimated to be 600 000.

Use the fact that $\dfrac{C}{P(C-P)} = \dfrac{1}{P} + \dfrac{1}{C-P}$ to show that the carrying capacity is approximately 1 130 000.

Further exercises

Ex 13D

- Q1-8, 12-15, 17
- (E) Other questions

2.3.3 Additional exercises

Source Haese et al. (2017, Ex 8H).

1. Consider the logistic differential equation $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{200}\right)$, $P(0) = 20$.

(a) Write P as a function of t.

(b) Find the value of P when $t = 10$.

(c) Discuss the behaviour of P as $t \to \infty$.

(d) Sketch the graph of P against t.

2. The population of koalas on an island is currently 500. Its growth rate is expected to be given by $\dfrac{dP}{dt} = 0.1P\left(1 - \dfrac{P}{3\,000}\right)$, where t is the time in years from now.

(a) Find the expected population after 8 years.

(b) Find the expected time taken for the population to increase to 2 000.

(c) What is the limiting population size?

(d) Sketch the graph of P against t.

3. In a small country town, rumours spread very fast. At 8 am on Monday, a rumour begins with 2 people. The number of people N who have heard the rumour grows according to the model

$$\frac{dN}{dt} = 0.8N\left(1 - \frac{N}{600}\right)$$

where t is the time in hours after 8 am.

(a) Write N as a function of t.

(b) How many people have heard the rumour by 11 am?

(c) How many people do you think live in the town?

(d) At what time would 500 people have heard the rumour?

4. There are 10^{30} molecules involved in a chemical reaction. Initially, 200 of the molecules are “active”, and any reaction between an “active” and an “inactive” molecule produces two “active” molecules. The nunber of “active” molecules grows according to the differential equation

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{10^{30}}\right)$$

where t is the time in seconds.

(a) Solve the differential equation, and hence write N in terms of k and t.

(b) Given that 1.5×10^7 molecules were “active” after 10^{-5} seconds, find k.

(c) At what time would you expect the reaction to be 99% complete?

5. 14 European foxes were released in Victoria in 1845 for sport hunting. Spreading rapidly out of control, the fox is now found throughout the mainland, except in the tropical northern regions. In 1900 there were 30 000 foxes in Australia, and today their population is steady at around 95 000.

(a) What features of the growth in the fox population suggest that a logistic model is appropriate?

(b) Suppose the population of foxes F grows according to the differential equation

$$\frac{dF}{dt} = kF\left(1 - \frac{F}{A}\right)$$

where t is the number of years since 1845.

i. State the value of A.

ii. Solve the differential equation, and use the information provided to write F in terms of t.

(c) Estimate the fox population in 1920.

(d) Estimate the time at which the fox population was:

i. 15 000 ii. 65 000

(e) Sketch the graph of F against t.

(f) When was the population growth rate a maximum? How does this appear on the graph of F against t?

Answers

1. (a) $P = \frac{200}{1+9e^{-0.2t}}$ (b) $P \approx 90.2$ (c) $t \to \infty$, $P \to 200$. (d) Check via technology. **2.** (a) $P = \frac{3\,000}{1+5e^{-0.1t}}$ (b) 924 (c) 23.0 years (d) 3 000 koalas (e) Check via technology. **3.** (a) $N = \frac{600}{1+299e^{-0.8t}}$ (b) 21 people (c) 600 people (d) 5:08 pm **4.** (a) $N \approx \frac{1\times10^{31}}{1+(5\times10^{27})e^{-kt}}$ (b) $k \approx 1.12 \times 10^{6}$ (c) after $\approx 6.09 \times 10^{-5}$ seconds **5.** (a) The population of foxes increased quickly at first, but later levelled off to approach a maximum. (b) i. $A = 95\,000$ ii. $F \approx \frac{95\,000}{1+\frac{94\,986}{14}e^{-0.146t}}$ (c) 85 100 foxes (d) i. 1894 ii. 1911 (e) Check via technology (f) In 1905, as it appears as an inflexion on the graph.

Section 3

Slope fields

Learning Goal(s)

Knowledge
Direction field

Skills
Sketch a graph on a field

Understanding
Particular solution

By the end of this section am I able to:

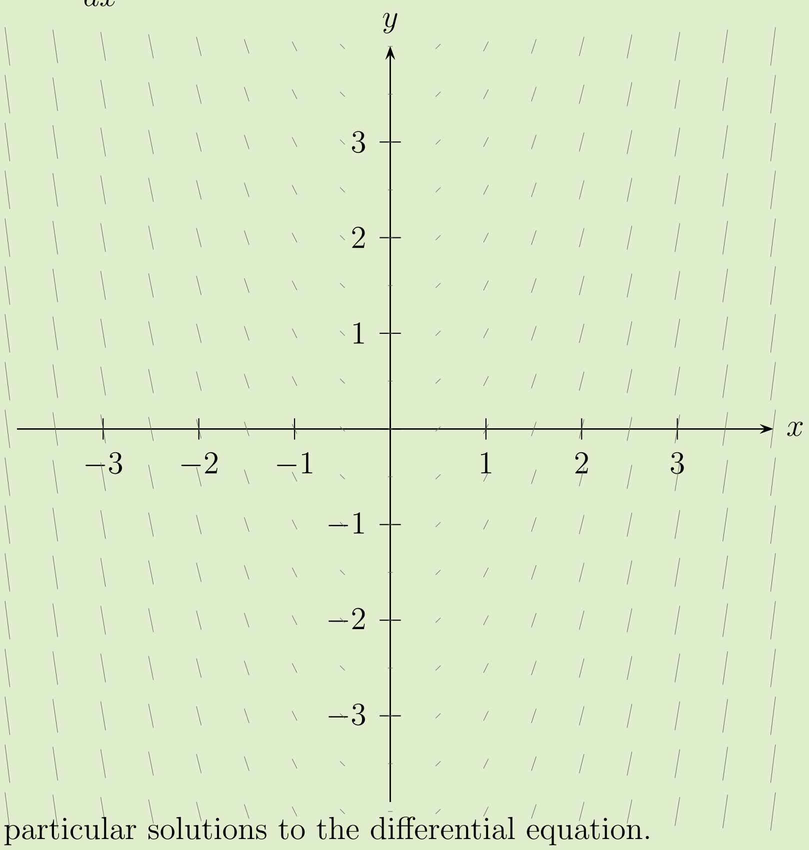
29.3 Sketch the graph of a particular solution given a direction field and initial conditions

Definition 8

The **slope/gradient/direction field** of the tangents to the solution curves represents ....gradients.... at many different grid points with short line segments.

Example 16

The slope field for $\frac{dy}{dx} = 2x$ is shown below.



Sketch a few particular solutions to the differential equation.

3.1 Constructing slope fields

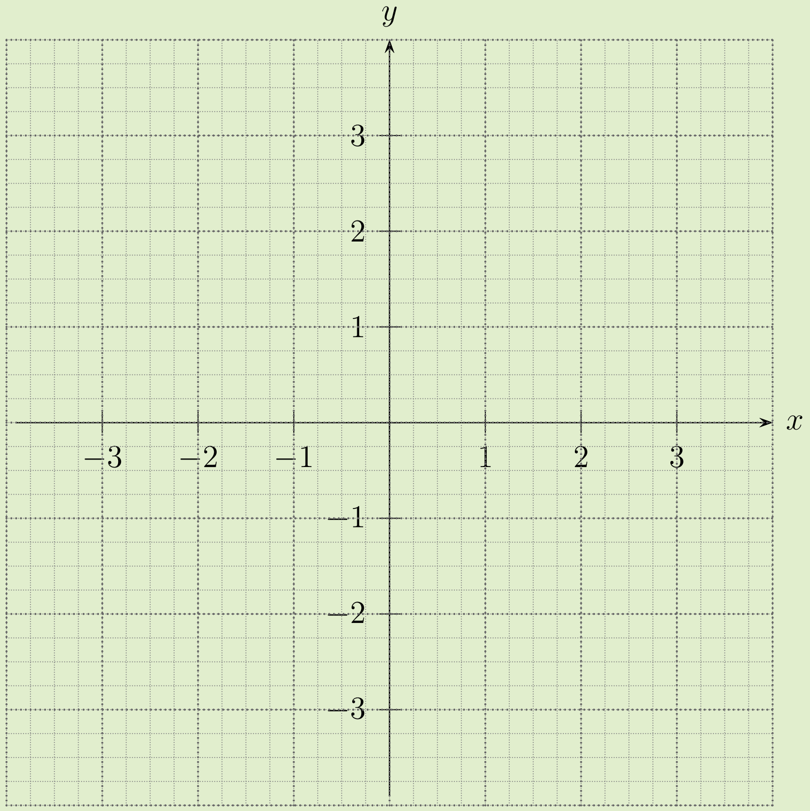
Steps

1. Fill out a table consisting of x, y and values of $\frac{dy}{dx}$ at the corresponding coordinates.
2. Plot the gradients of the tangents at the appropriate coordinate.

Example 17

Fill in the following table with the relevant gradients at the points indicated to construct the slope field for $\frac{dy}{dx} = 2x$.

y/x	-4	-3	-2	-1	0	1	2	3	4
4									
3									
2									
1									
0									
-1									
-2									
-3									
-4									



Example 18

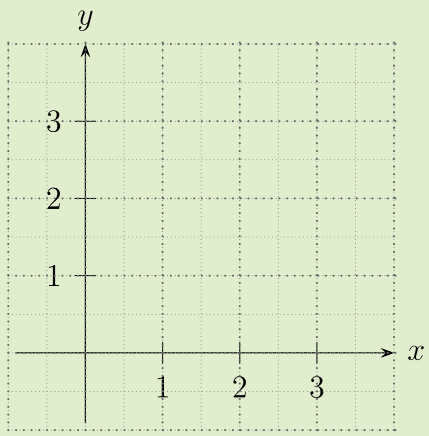
[Section 8F] (Haese et al., 2017, p.246) Consider the differential equation $\dfrac{dy}{dx} = xy$.

(a) Construct the slope field for the differential equation using the integer grid points for x, $y \in [0, 4]$.

(b) Find the equation of the particular solution curve which passes through $(2, 1)$.

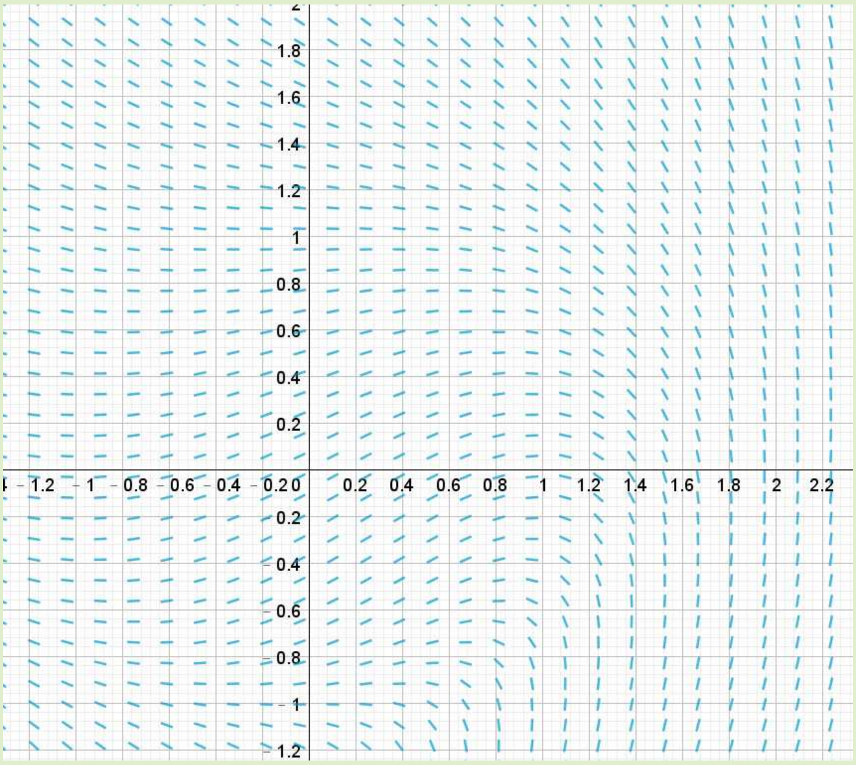
(c) Sketch the solution curve from the previous part on the slope field.

y/x	0	1	2	3	4
4					
3					
2					
1					
0					



Example 19

[Section 8F] (Haese et al., 2017, p.245) The slope field for $\dfrac{dy}{dx} = \dfrac{1 - x^2 - y^2}{y - x + 2}$ is shown.



(a) Find the gradient to the tangent to the solution curve at $(1, 1)$.

(b) Sketch the solution curve which passes through $(1, 1)$.

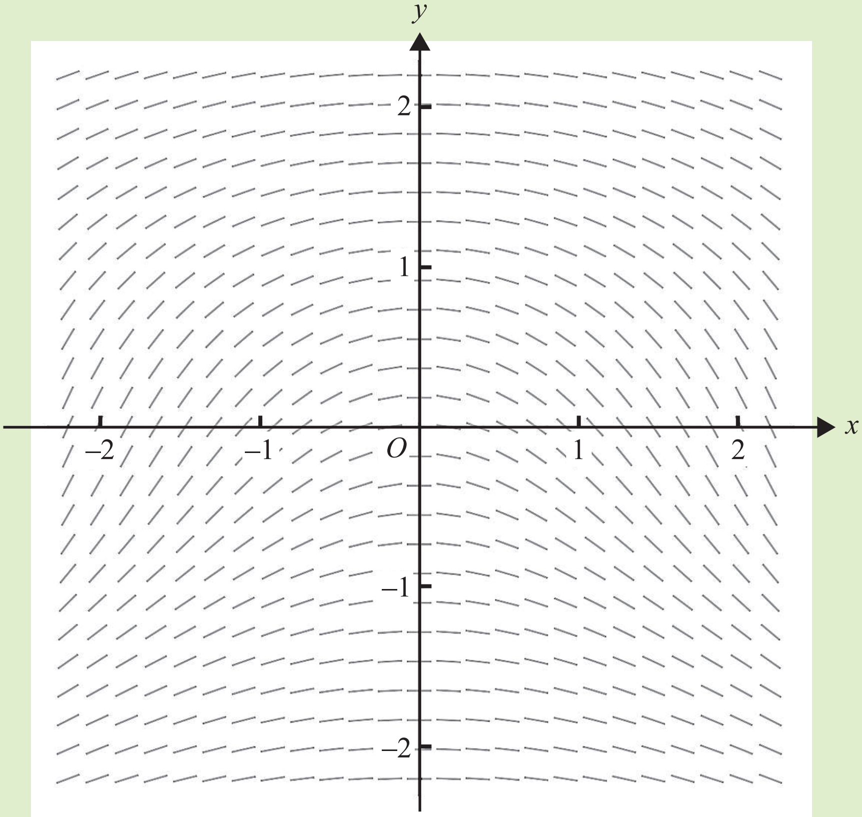
Answer: $m = -\frac{1}{2}$

Example 20

[2017 VCE Specialist Mathematics Paper 1 Q8] A slope field representing the differential equation

$$\frac{dy}{dx} = \frac{-x}{1+y^2}$$

is shown below.



(a) Sketch the solution curve of the differential equation corresponding to the condition $y(-1) = 1$ on the slope field above and, hence, estimate the positive value of x when $y = 0$. **2**

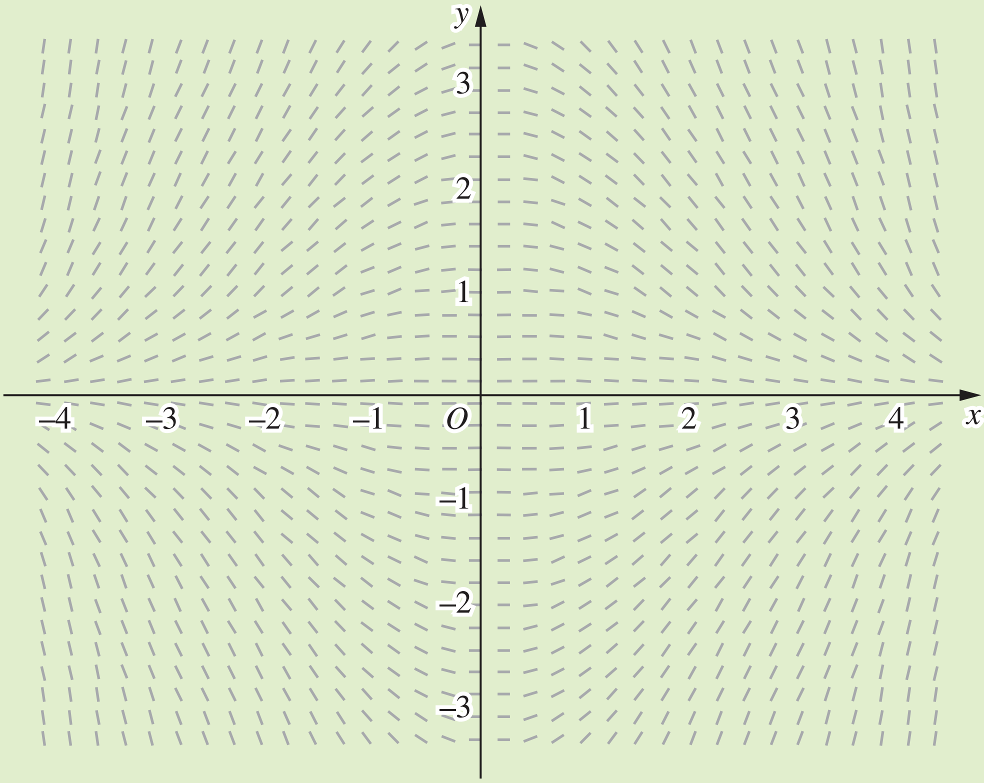
Give your answer correct to one decimal place.

(b) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{-x}{1+y^2}$ with the condition $y(-1) = 1$. Express your answer in the form $ay^3 + by + cx^2 + d = 0$ where a, b, c and d are integers. **2**

3.2 Interpreting slope fields

Example 21

[2020 Ext 1 HSC Sample Q5] The slope field for a first order differential equation is shown.

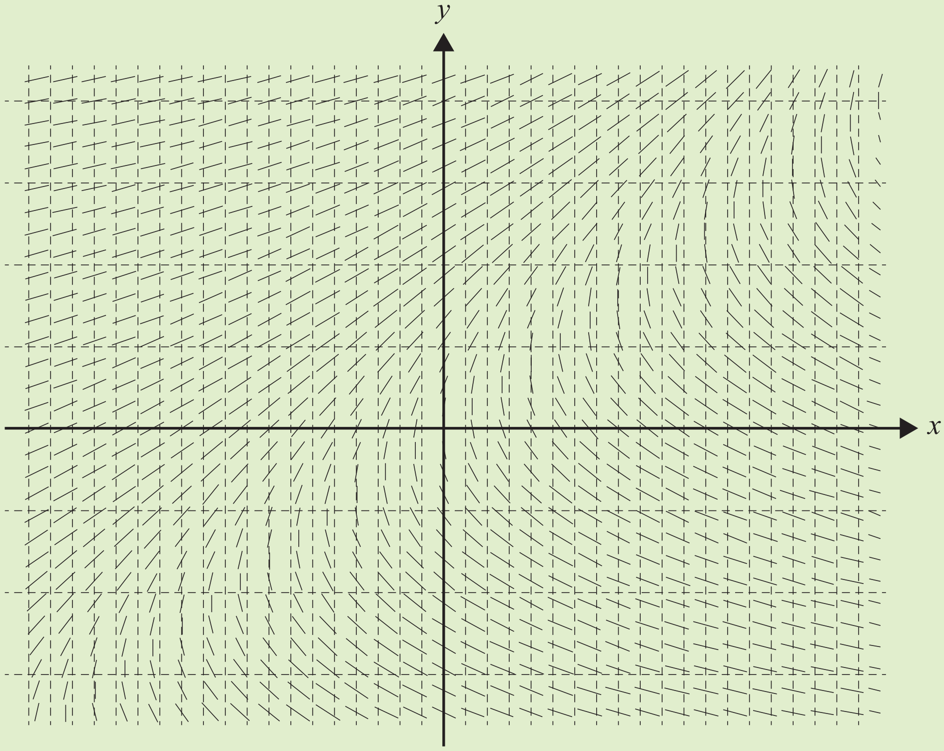


Which of the following could be the differential equation represented?

(A) $\dfrac{dy}{dx} = \dfrac{x}{3y}$ (B) $\dfrac{dy}{dx} = -\dfrac{x}{3y}$ (C) $\dfrac{dy}{dx} = \dfrac{xy}{3}$ (D) $\dfrac{dy}{dx} = -\dfrac{xy}{3}$

Example 22

[2014 VCE Specialist Mathematics Paper 2 Q14] The differential equation that is best represented by the above direction field is



(A) $\dfrac{dy}{dx} = \dfrac{1}{x-y}$

(B) $\dfrac{dy}{dx} = y - x$

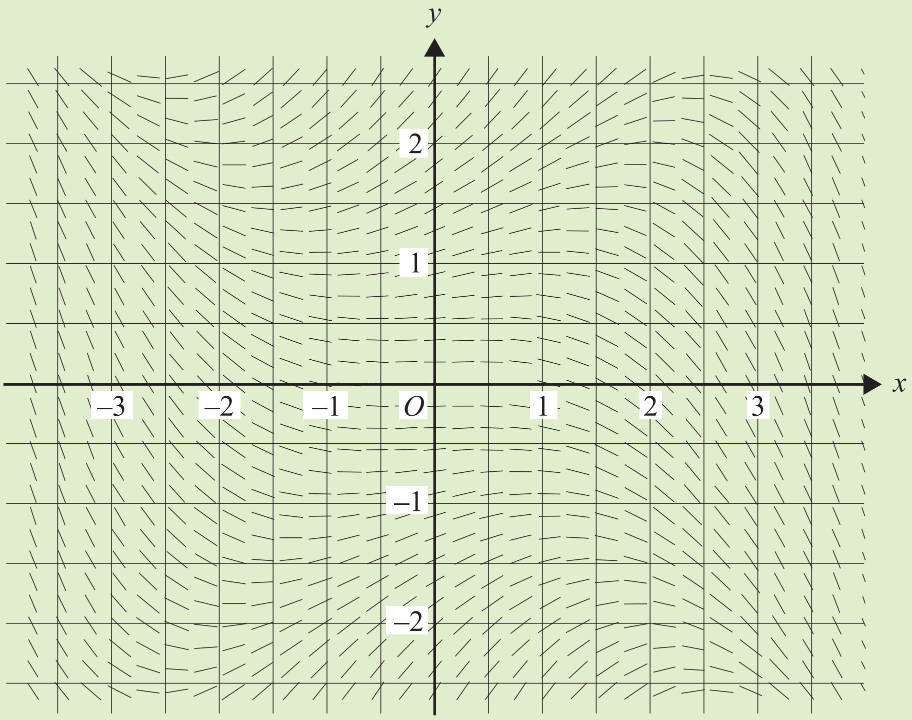
(C) $\dfrac{dy}{dx} = \dfrac{1}{y-x}$

(D) $\dfrac{dy}{dx} = x - y$

(E) $\dfrac{dy}{dx} = \dfrac{1}{y+x}$

Example 23

[2015 VCE Specialist Mathematics Paper 2 Q13] The direction field for a certain differential equation is shown.

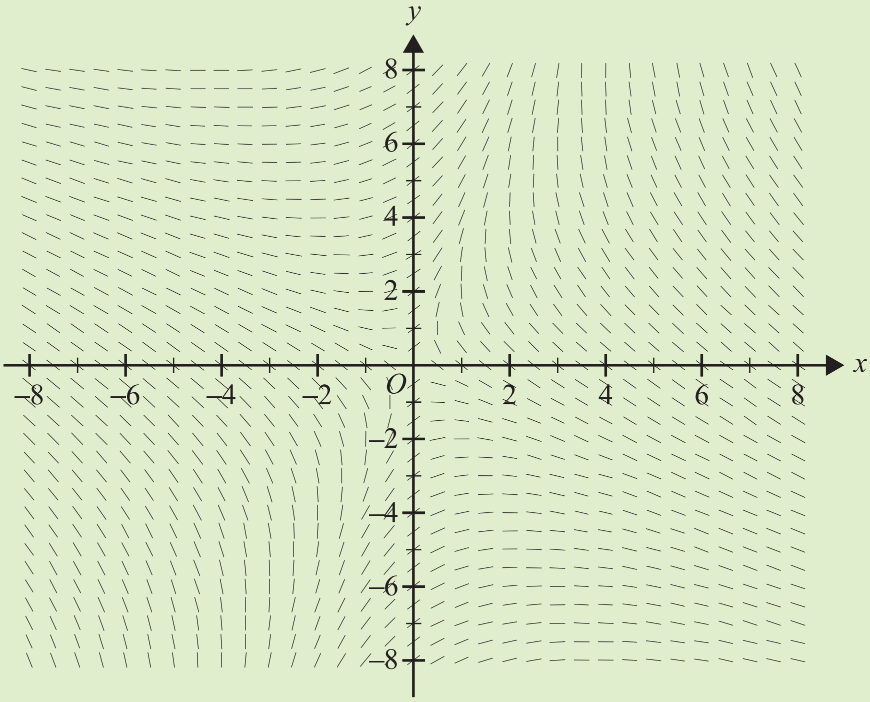


The solution curve to the differential equation that passes through the point $(2.5, 1.5)$ could also pass through:

(A) $(0, 2)$ (B) $(1, 2)$ (C) $(3, 1)$ (D) $(3, -0.5)$ (E) $(-0.5, 2)$

Example 24

[2018 VCE Specialist Mathematics Paper 2 Q10] The differential equation that best represents the direction field below is:



(A) $\dfrac{dy}{dx} = \dfrac{2x+y}{y-2x}$

(B) $\dfrac{dy}{dx} = \dfrac{x+2y}{2x-y}$

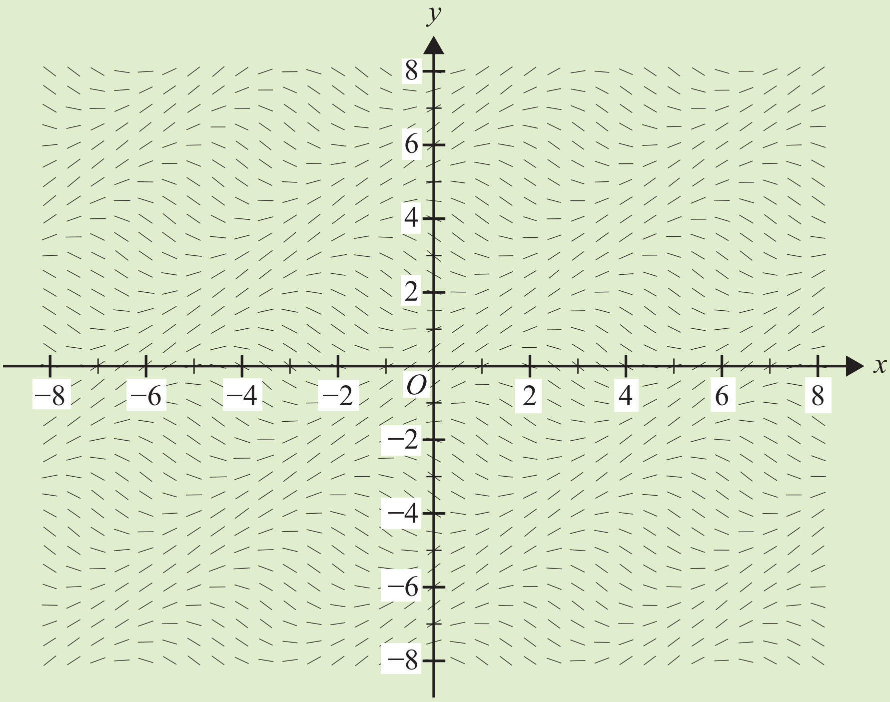
(C) $\dfrac{dy}{dx} = \dfrac{2x-y}{x+2y}$

(D) $\dfrac{dy}{dx} = \dfrac{x-2y}{y-2x}$

(E) $\dfrac{dy}{dx} = \dfrac{2x+y}{2y-x}$

Example 25

[2019 VCE Specialist Mathematics Paper 2 Q9] The differential equation that has the diagram below as its direction field is:



(A) $\dfrac{dy}{dx} = \sin(y - x)$

(B) $\dfrac{dy}{dx} = \cos(y - x)$

(C) $\dfrac{dy}{dx} = \sin(x - y)$

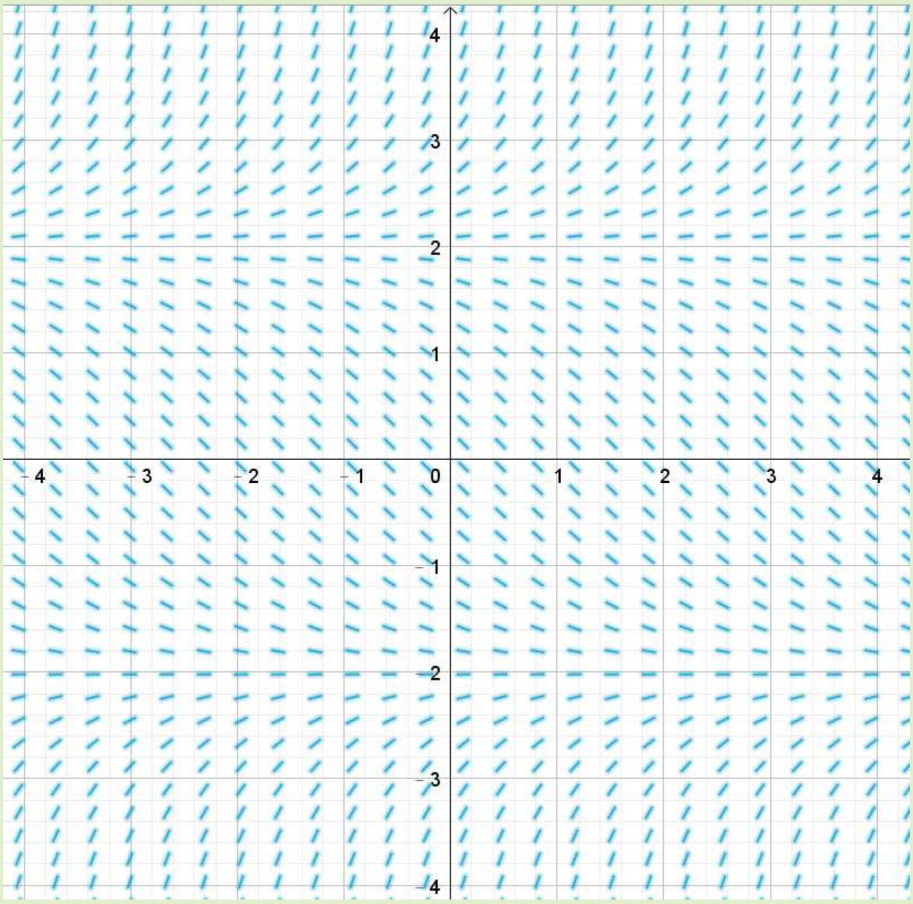
(D) $\dfrac{dy}{dx} = \dfrac{1}{\cos(y - x)}$

(E) $\dfrac{dy}{dx} = \dfrac{1}{\sin(y - x)}$

Example 26

(a) Sketch possible solution curves to the slope field to the differential equation

$$\frac{dy}{dx} = \frac{1}{4}(y-2)(y+2)$$



(b) From the slope field, identity the constant solutions, that is, the equilibrium solutions.

(c) Substitute into the DE to show that they are solutions.

(d) If the horizontal axis is time, describe the behaviour of the solution curves near those constant solutions, and distinguish between them.

Important note

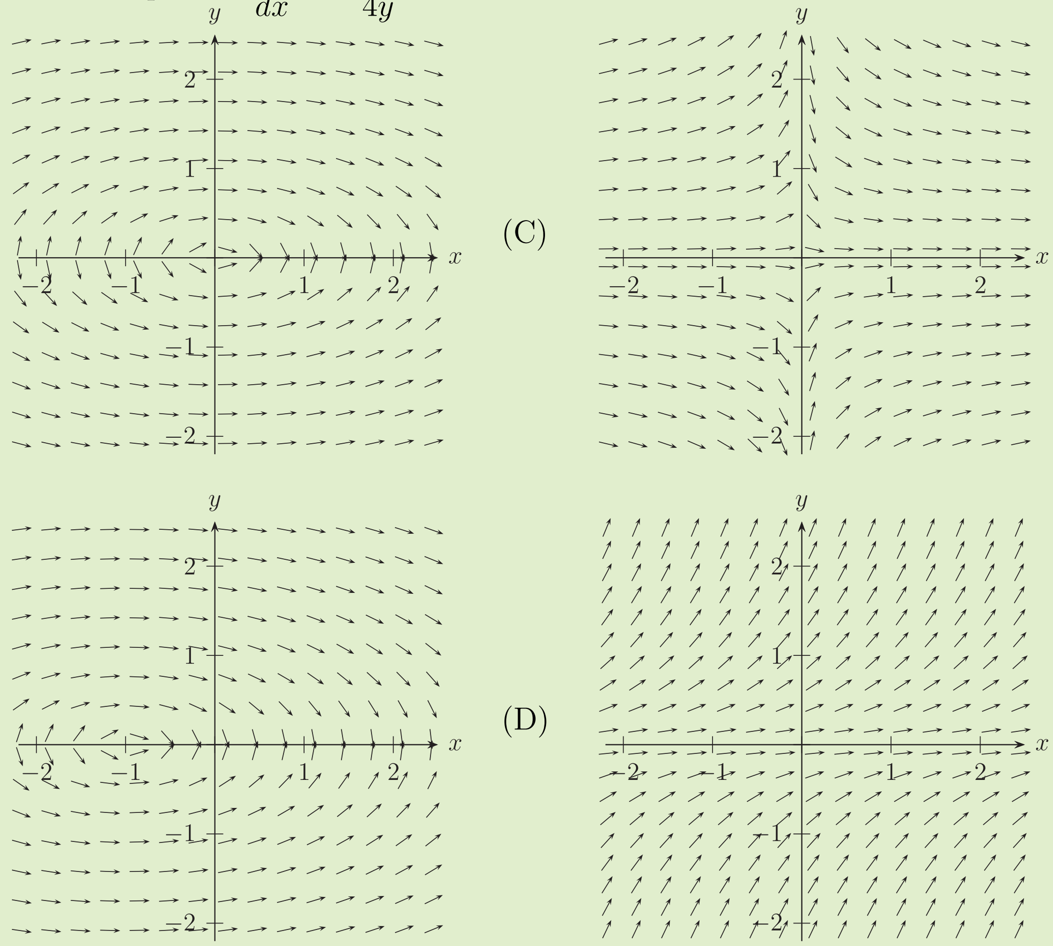
Horizontal solutions are asymptotes

Example 27

[2020 Ext 1 HSC Q7] Which of the following best represents the direction field for the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{4y}$?

(A) (C)

(B) (D)



Further exercises

Ex 13B

- Q5-15

Ex 13D

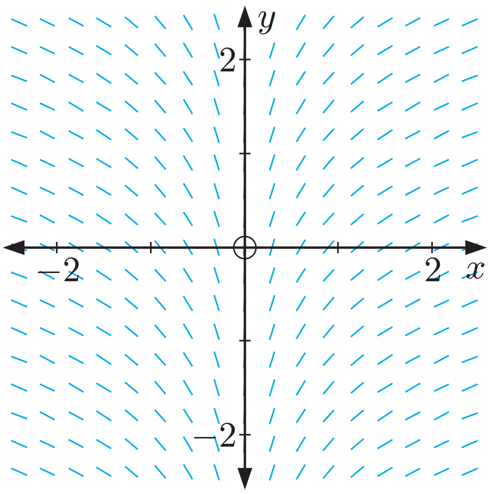
- Q10, 11, 15, 16

3.2.1 Additional exercises

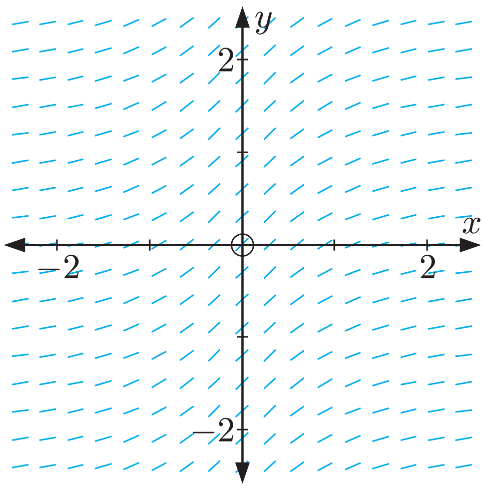
Source (Haese et al., 2017, Ex 8F)

1. Slope fields for two differential equations are plotted below for $x, y \in [-2, 2]$. In each case, use the slope field to graph the solution curve passing through $(1, 1)$.

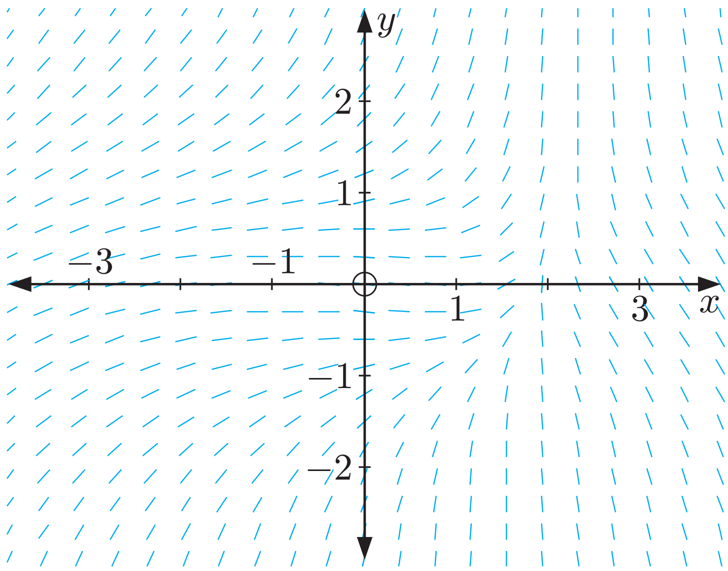
(a)



(b)



2. The slope field for the differential equation $\dfrac{dy}{dx} = \dfrac{-1 + x^2 + 4y^2}{y - 5x + 10}$ is shown.



(a) Find the gradient of the tangent to the solution curve at the origin.

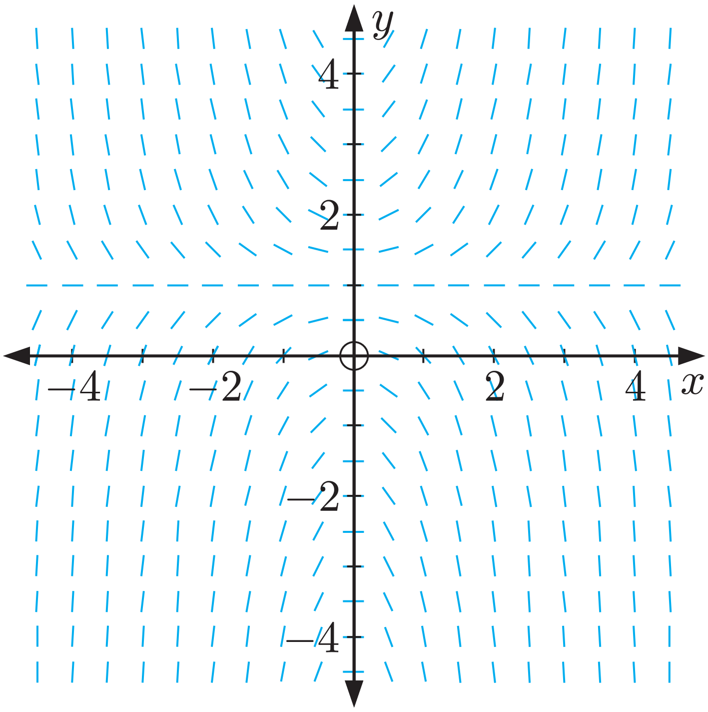
. . . .

. . . .

. . . .

(b) Sketch the particular solution passing through the origin.

3. The slope field for the differential equation $\dfrac{dy}{dx} = x(y-1)$ is shown.



(a) Sketch the solution curve which passes through $(0, 2)$.

(b) Find the equation of the solution curve drawn in (a).

. . . .

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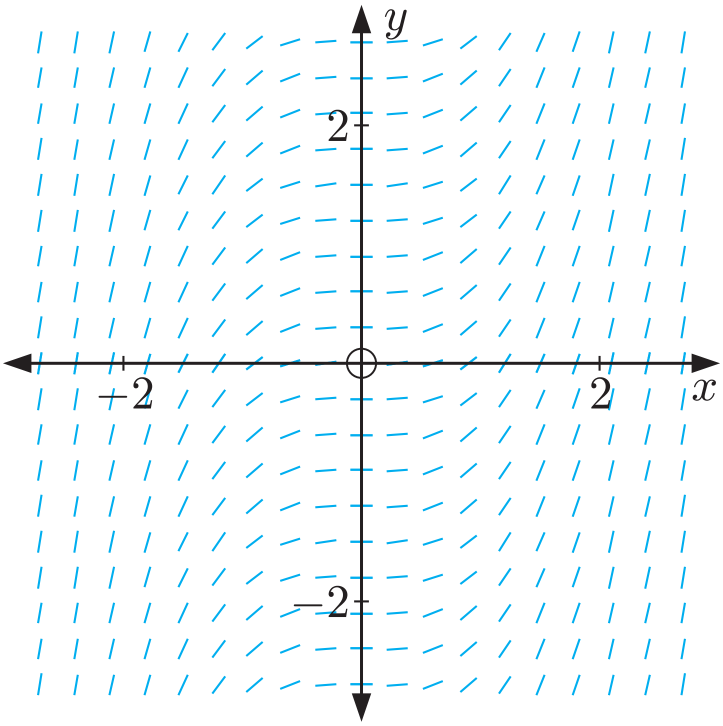
4. Consider the slope field for the differential equation $\frac{dy}{dx} = x^2$.



(a) Show that a general solution for the differential equation is $y = \frac{1}{3}x^3 + C$.

. . . .

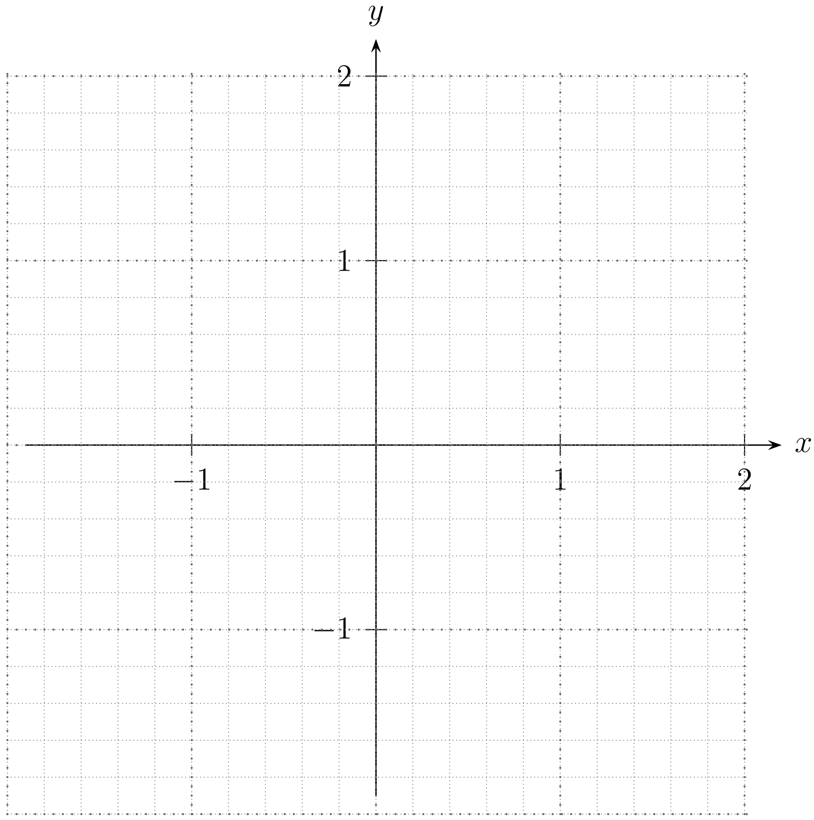
. . . .

(b) Sketch the particular solution curve for

i. $C = 1$

ii. $C = 2$

5. (a) Construct the slope field for the differential equation $\dfrac{dy}{dx} = \dfrac{1}{2}xy$ using integer grid points $x, y \in [-2, 2]$.



(b) Find the equation of the particular solution curve which passes through $(1, -1)$. Sketch this curve on your slope field.

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. . . .

. . . .

. . . .

Section 4

Applications and problem solving

Learning Goal(s)

Knowledge
Logistic curve

Skills
Solve

Understanding
Chemistry/Biology/Economics phenomena modelled by the logistic curve

By the end of this section am I able to:

29.6 Model and solve differential equations including to the logistic equation that will arise in situations where rates are involved, for example in chemistry, biology and economics

Example 28

[**2006 VCE Specialist Mathematics Paper 2 Q10**] A chemical dissolves in a pool at a rate equal to 5% of the amount of **undissolved** chemical. Initially the amount of undissolved chemical is 8 kg and after t hours x kilograms has dissolved.

The differential equation which models this process is

(A) $\dfrac{dx}{dt} = \dfrac{x}{20}$

(B) $\dfrac{dx}{dt} = \dfrac{8-x}{20}$

(C) $\dfrac{dx}{dt} = \dfrac{x-8}{20}$

(D) $\dfrac{dx}{dt} = -\dfrac{x}{20}$

(E) $\dfrac{dx}{dt} = 8 - \dfrac{x}{20}$

Example 29

[2007 VCE Specialist Mathematics Paper 2 Q14] The rate at which a type of bird flu spreads throughout a population of 1 000 birds in a certain area is proportional to the product of the number N of infected birds and the number of birds still not infected after t days. Initially two birds in the population are found to be infected.

A differential equation, the solution of which models the number of infected birds after t days, is

(A) $\dfrac{dN}{dt} = k\dfrac{1\,000 - N}{1\,000}$

(B) $\dfrac{dN}{dt} = k(N-2)(1\,000 - N)$

(C) $\dfrac{dN}{dt} = kN(1\,000 - N)$

(D) $\dfrac{dN}{dt} = kN(1\,000 - (N+2))$

(E) $\dfrac{dN}{dt} = k(N+2)(1\,000 - N)$

Example 30

[2008 VCE Specialist Mathematics Paper 2 Q14] The volume of water V m^3 in a cylindrical tank when it is filled to a depth of h metres is given by $V = 4h$. Water flows into the tank at a rate of 0.2 m^3 per minute and leaks out at a rate of $0.01\sqrt{h}$ m^3 per minute. The differential equation, which when solved would enable h to be expressed in terms of t, is

(A) $\dfrac{dh}{dt} = 0.2 - 0.01\sqrt{h}$

(B) $\dfrac{dh}{dt} = 4\left(0.2 - 0.01\sqrt{h}\right)$

(C) $\dfrac{dh}{dt} = \dfrac{20 - \sqrt{h}}{400}$

(D) $\dfrac{dh}{dt} = \dfrac{40}{20 - \sqrt{h}}$

(E) $\dfrac{dh}{dt} = 20 - \dfrac{400}{\sqrt{h}}$

Example 31

[2010 VCE Specialist Mathematics Paper 1 Q7] (X2) Consider the differential equation

$$\frac{d^2y}{dx^2} = \frac{4x}{\left(1-x^2\right)^2} \qquad -1 < x < 1$$

for which $\dfrac{dy}{dx} = 3$ when $x = 0$, and $y = 4$ when $x = 0$.

Given that $\dfrac{d}{dx}\left(\dfrac{2}{1-x^2}\right) = \dfrac{4x}{\left(1-x^2\right)^2}$, find the solution of this differential equation.

Answer: $y = x + \ln\left(\frac{1+x}{1-x}\right) + 4$

Example 32

[2016 VCE Specialist Mathematics Paper 2 Q3] A tank initially has 20 kg of salt dissolved in 100 L of water. Pure water flows into the tank at a rate of 10 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 5 L/min.

If x kilograms is the amount of salt in the tank after t minutes, it can be shown that the differential equation relating x and t is

$$\frac{dx}{dt} + \frac{x}{20+t} = 0$$

(a) Solve this differential equation to find x in terms of t. **3**

A second tank initially has 15 kg of salt dissolved in 100 L of water. A solution of $\frac{1}{60}$ kg of salt per litre flows into the tank at a rate of 20 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 10 L/min.

(b) If y kilograms is the amount of salt in the tank after t minutes, write down an expression for the concentration, in kg/L, of salt in the second tank at time t. **1**

(c) Show that the differential equation relating y and t is $\frac{dy}{dt} + \frac{y}{10+t} = \frac{1}{3}$. **2**

(d) Verify by differentiation and substitution into the left side that $y = \frac{t^2+20t+900}{6(10+t)}$ satisfies the **differential equation in part (c)**. Verify that the given solution for y also satisfies the **initial condition**. **3**

(e) Find when the concentration of salt in the second tank reaches 0.095 kg/L. Give your answer in minutes, correct to two decimal places. **2**

Answer: (a) $x = \frac{400}{20+t}$ (b) $\frac{y}{100+10t}$ (c) Show (d) Verify (e) $t = 3.05$

Further exercises

Ex 13E

- Q1-13

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

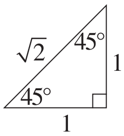
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

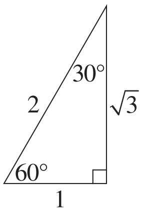
$A = \frac{1}{2}ab\sin C$



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \frac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

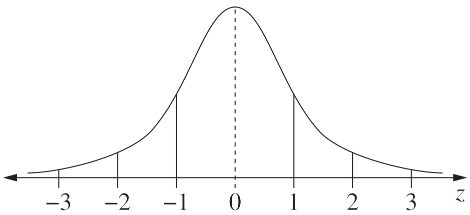
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

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